Investigating stable Fixed Point Theorems for Weakly Compatible Mappings in Complete Intuitionistic Fuzzy Metric Spaces

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ABSTRACT

Several frequent stable point theorems are formulated using perfect intuitionistic fuzzy metric (IFM) spaces and organized in this article. The purpose of this article is to validate these theorems under conditions that increase their future applicability. The concept of reciprocally continuous mappings functions as a requirement for two functions to stay mutually continuous together while weakly compatible mappings need supplementary conditions to determine fixed points. The findings achieve better strength because the methodology incorporates associated sequences. The study contributes to intuitionistic fuzzy metric (IFM) spaces fixed point theory by implementing various mathematical methods. The paper has specific aims to advance current findings through an exchange of complete metric spaces with complete IFM spaces.

Keywords: Complete Intuitionistic fuzzy metric space (IFM), stable Fixed Point, Self-mapping, Weakly compatible, Reciprocally continuous, Associated sequence.

1) Introduction

The values of decision-making variables within human reasoning exist as fuzzy sets which enable an approach that gives flexibility and natural insight for decision processes. The use of fuzzy rules for system behaviour descriptions lowers requirements for precise data collection along with precise data processing. Data compression happens through this method because simplified complex information maintains valuable meaning. Some applications require more detailed system description than what fuzzy linguistic variables with their associated membership functions can provide.

ISSN: 0012-0413
2025 123(4)

System analysis benefits from a conceptual shift because linguistic variables enable practical human approaches to represent imprecision and uncertainty across multiple domains.

Zadeh [1] introduced fuzzy sets in 1965 while Atanassov [2] established IFS as the full version of FS in 1986.

The interpretation demonstrates variable logical membership values instead of defined truth levels. The enriched modeling system enhances decision-related uncertainty management when applied to both organizational decisionology and numerical modeling operations. Intuitionistic fuzzy sets need logical values that fulfill the requirement $\gamma A(x)+\mu A(x)\geq 1$ which demonstrates the relationship between membership degree and non-membership degree denoted by $\gamma A(x)$ and $\mu A(x)$. All results can be extended from fuzzy sets to intuitionistic fuzzy sets even though this method does not function in the opposite direction. The theory of intuitionistic fuzzy sets provides an efficient tool for modelling both element decision and hesitancy to belong to a set. Traditional fuzzy set theory determines the non-membership value automatically from the [0,1] membership range yet intuitionistic fuzzy sets provide a more flexible framework which benefits uncertain and hesitant systems. According to intuitionistic fuzzy set theory the maximum value of non-membership boundary should be set at $1-a$.

The management of uncertain and poorly known set elements becomes clearer through IFS in such situations. The development of fuzzy image processing remains slow because researchers have published only few methods.

The precision of information retrieval enhances through intuitionistic fuzzy set theory since it properly represents hesitation patterns found in real-world systems. Sets represented by intuitionistic fuzzy theory contain an added freedom degree which makes them outperform traditional fuzzy sets when used in complex decision processes. Coker [3] focuses on intuitionistic fuzzy topological spaces as its main research area. The research of fixed points takes place within the context of IFM spaces according to the work of Alaca et al[4].

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The paper by Turkoglu et al. [5] presents different conclusions regarding stable fixed point theorems in IFM. Different approaches to establish fuzzy metrics have been presented by various scholars [(6, 7 , 8)]. The study conducted by Grabiec [9] builds on the foundational work of Banach and Edelstein fixed point theorems, which focus on contractive mappings. These theorems are particularly relevant when applied to complete and compact fuzzy metric spaces, as defined by Kramosil and Michalek [10]. The research by George et.al., [(11, 12)] introduced a modification to the original fuzzy metric space definition provided by Kramosil et.al., incorporating a Hausdorff topology.

Following this, several authors [(13, 14, 15, 16, 17)] have advanced the field by establishing various common fixed point results. Their focus has primarily been on mappings that satisfy weakly commuting and R-weakly commuting conditions within the framework of fuzzy metric spaces, often under more general and diverse settings. These results are significant because they expand the applicability of fixed point theory to include mappings that do not necessarily adhere to strict commutativity, thus providing a more comprehensive understanding of the behavior of mappings in fuzzy metric spaces.

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J. H. Park.et.al., [18] introduced Intuitionistic Fuzzy Metric (IFM) spaces, which significantly expanded the scope of fixed point theory in FM spaces. Building on this, R. P. Pant ([19, 20, 21, 22]) developed multiple stable fixed point theorems for contractive mathematical function and mappings that are non-compatible. His research also extended to include stable fixed points of noncommuting mappings and Lipschitz-type mapping pairs. Additionally, V. Pant [23] contributed by establishing fixed point theorems within fuzzy metric spaces, and H. K. Pathak et al. [24] focused on stable fixed point theorems for R-weakly transmuting mappings. This paper takes a different approach by utilizing reciprocally continuous and compatible mappings, alongside weakly compatible and associated sequences, to prove common fixed point results within the framework of complete IFM spaces. The primary objective of this research is to provide generalized scores that extend the theory of fixed points to complete intuitionistic fuzzy metric structures, thus enriching the existing body of work in this area.$(W, K, L, *, \diamond)$

2) Preliminaries

Definition 2.1. Let two self-functions, F and G, operate on an IFM space, denoted as These functions are declared to be uncertainly commuting if the resulting term holds for all a∈W$K(FGa, GFa, kt) \geq K(Fa, Ga, t)$ and $L(FGa, GFa, kt) \leq L(Fa, Ga, t)$ for all $a \in W$.

Definition 2.2. A combination of functions F and G in an IFM space$(W, K, L, *, \diamond)$ is called matching if $\lim_{m\to\infty} K(FGa_m, GFa_m, kt) = 1$ and $\lim_{m\to\infty} L(FGa_m, GFa_m, kt) = 0$ for all $t > 0$, when $\{a_m\}$ is a series in W where $\lim_{m\to\infty} Fa_m = \lim_{m\to\infty} Ga_m = v$ for several $v \in W$.

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The concepts of commuting mapping and weak commuting mapping show an inclusion relationship with one direction being obvious while the other direction remains uncertain.

Definition 2.3. In the context of weak compatibility, two self-maps, F and G, on an (IFM) space are said to produce commuting results at their coincidence point. This means that if for some point a in the space, F(a)=G(a), then the actions of F and G on the space are inadequately corresponding, and they commute at this point. Specifically, if two mappings F and G satisfy this situation for some point a, then the mappings are inadequately corresponding. This ensures that their behavior is consistent at the coincidence point, even if they are not necessarily strongly compatible. It is important to note that while weak compatibility guarantees commuting at a coincidence point, this does not imply that weak compatibility is equivalent to strong compatibility. The proof of weak compatibility holds despite the refutation of the claim that weak compatibility implies strong compatibility.

Definition 2.4 introduces the concept of two mappings, F and G, as "equally constant" on an IFM space. Two mappings are called equally constant if their behavior remains unchanged under certain conditions, typically defined by specific criteria within the space. In other words, the mappings F and G maintain constant outputs or exhibit similar fixed-point behavior under these conditions, reinforcing the consistency in their interaction within the IFM framework $FGv_m \to Fz$, $GFv_m \to Gz$, whenever $\{v_m\}$ is a sequence such that $Fv_m \to z, Gv_m \to z$ for some z in W.

The connection of F and G together leads to reciprocally continuous behavior but the reverse condition is invalid.

Theorem 2.5. Four self-functions F, G, Y and Z operate on complete intuitionistic fuzzy metric spaces which fulfill the following restrictions

$$F(W) \subset Z(W) \text{ and } G(W) \subset Y(W) \tag{1}$$

$$K(Fa, Gb, kt) \geq \alpha \frac{K(Zb, Gb, t)[1+K(Ya, Fa, t)]}{[1+K(Ya, Zb, t)]} + \beta K(Ya, Zb, t)$$

$$L(Fa, Gb, kt) \leq \alpha \frac{L(Zb, Gb, t)[1+L(Ya, Fa, t)]}{[1+L(Ya, Zb, t)]} + \beta L(Ya, Zb, t) \tag{2}$$

for all a, b in X where $\alpha, \beta \geq 0, \alpha + \beta < 1$.

One of *Y, Z, F* and *G* is continuous (3)

Pairs (F, Y) and (G, Z) are consistent on X (4)

then *Y, Z, F* and *G* have a exclusive stable fixed point in *W*.

Associated sequence 2.6. The four mappings Y, Z, F and G operate on the IFM space under the restriction (1). We define the sequence via this association method to any point with where is part of W. This sequence will be called “Associated sequence of” due to its dependence on the four self-maps Y, Z, F and G.

 ISSN: 0012-0413

Lemma 2.7. A complete IFM space contains Y, Z, F and G as function that fulfill the respective conditions (1) and (2). The sequence derived from four self-maps Y, Z, F and G produces a Cauchy sequence within W.

To prove the statement we apply the definition of associated sequence (2.6).

$$K(b_{2m},b_{2m+1},kt)=K(Fa_{2m},Ga_{2m+1},kt)\geq \alpha\frac{K(Za_{2m+1},Ga_{2m+1},t)[1+K(Ya_{2m},Fa_{2m},t)]}{[1+K(Ya_{2m},Zb_{2m+1},t)]}+\beta K(Ya_{2m},Zb_{2m+1},t)$$

$$=\alpha\frac{K(b_{2m},b_{2m+1},t)[1+K(b_{2m-1},b_{2m},t)]}{[1+K(b_{2m-1},b_{2m},t)]}+\beta K(b_{2m-1},b_{2m},t)$$

$$K(b_{2m},b_{2m+1},kt)=\alpha K(b_{2m},b_{2m+1},t)+\beta K(b_{2m-1},b_{2m},t)$$

$$(1-\alpha)K(b_{2m},b_{2m+1},t)\geq \beta K(b_{2m-1},b_{2m},t)$$

$$K(b_{2m},b_{2m+1},t)\geq \frac{\beta}{(1-\alpha)}K(b_{2m-1},b_{2m},t)$$

$$L(b_{2m},b_{2m+1},kt)=L(Fa_{2m},Ga_{2m+1},kt)\leq \alpha\frac{L(Za_{2m+1},Ga_{2m+1},t)[1+L(Ya_{2m},Fa_{2m},t)]}{[1+L(Yx_{2n},Zy_{2n+1},t)]}+\beta L(Ya_{2m},Zb_{2m+1},t)$$

$$=\alpha\frac{L(b_{2m},b_{2m+1},t)[1+L(b_{2m-1},b_{2m},t)]}{[1+L(b_{2m-1},b_{2m},t)]}+\beta L(b_{2m-1},b_{2m},t)$$

$$L(b_{2m},b_{2m+1},kt)=\alpha L(b_{2m},b_{2m+1},t)+\beta L(b_{2m-1},b_{2m},t)$$

$$(1-\alpha)L(b_{2m},b_{2m+1},t)\leq \beta L(b_{2m-1},b_{2m},t)$$

$$L(b_{2m},b_{2m+1},t)\leq \frac{\beta}{(1-\alpha)}L(b_{2m-1},b_{2m},t)$$

$K(b_{2m},b_{2m+1},t)\geq hK(b_{2m-1},b_{2m},t)$, $L(b_{2m},b_{2m+1},t)\leq L(b_{2m-1},b_{2m},t)$ where $h=\frac{\beta}{(1-\alpha)}$

Now

$$K(b_m,b_{m+1},t)\geq hK(b_{m-1},b_m,t)\geq h^2K(b_{m-2},b_{m-1},t)\geq \ldots\ldots\geq h^mK(b_0,b_1,t)$$

$$L(b_m,b_{m+1},t)\leq hL(b_{m-1},b_m,t)\leq h^2L(b_{m-2},b_{m-1},t)\leq \ldots\ldots\leq h^mL(b_0,b_1,t)$$

For every integer $p>0,$ we get

$$K(b_m,b_{m+p},t)\geq K(b_m,b_{m+1},t)+K(b_{m+1},b_{m+2},t)+\ldots\ldots+K(b_{m+p-1},b_{m+p},t)$$

$$\geq h^mK(b_0,b_1,t)+h^{m+1}K(b_0,b_1,t)+\ldots\ldots\ldots+h^{m+p-1}K(b_0,b_1,t)$$

$$\geq (h^m+h^{m+1}+\ldots\ldots\ldots+h^{m+p-1})K(b_0,b_1,t)$$

$$\geq h^m(1+h+h^2+\ldots\ldots+h^{p-1})K(b_0,b_1,t)$$

$$L(b_m,b_{m+p},t)\le L(b_m,b_{n+1},t)+L(b_{m+1},b_{m+2},t)+\ldots\ldots+L(b_{m+p-1},b_{m+p},t)$$
$$\le h^m L(b_0,b_1,t)+h^{m+1}L(b_0,b_1,t)+\ldots\ldots\ldots+h^{m+p-1}L(b_0,b_1,t)$$
$$\le (h^m+h^{m+1}+\ldots\ldots\ldots+h^{m+p-1})L(b_0,b_1,t)$$
$$\le h^m(1+h+h^2+\ldots\ldots+h^{p-1})L(b_0,b_1,t)$$

Since $h<1,\ h^m\to 0$ as $m\to\infty$, so that $K(b_m,b_{m+p},t)\to 0, L(b_m,b_{m+p},t)\to 0$. By showing that The sequence converges to a point W, and the Cauchy sequence property in W, along with the completeness of W, ensures that the sequence has a limit, which equals W. However, the converse condition does not necessarily hold true for self-maps Y, Z, F, and G in a complete intuitionistic fuzzy metric space. Specifically, even if the conditions (1) and (2) are satisfied, and the associated sequence converges, the space does not have to be complete. This highlights that the completeness of the space does not guarantee the existence of limits for all sequences, indicating a limitation of the space's properties in certain cases.

3) Main Result

Theorem 3.1. The self-maps Y, Z, F, G operate on a complete IFM space which fulfills conditions (1) and (2) and the specified conditions.

The pairs are reciprocally continuous and compatible and the pairs exhibit weak compatibility. (5)

compatible (5)

The sequence of maps Y, Z, F and G satisfy the provided definitions concerning four self-maps when one considers the associated sequence**.**

$Fa_0, Ga_1, Fa_2, Ga_3,\ldots\ldots\ldots Fa_{2n}, Ga_{2n+1},\ldots\ldots$ converges to $z\in W$ as $m\to\infty$ (6)

then *Y, Z, F* and *G* have a exclusive stable fixed point z in W.

Proof. From the condition (6), $Fa_0, Ga_1, Fa_2, Ga_3,\ldots\ldots\ldots Fa_{2n}, Ga_{2n+1},\ldots\ldots$ converges to $z\in W$ as $m\to\infty$

First assume that the pair (F,Y) is reciprocally permanent and compatible, then from the definition of reciprocally continuity of (F,Y) if $Fa_{2m}\to z$, $Ya_{2m}\to z$ as then

$$Fbx_{2m}\to Fz\ ,\ YFa_{2m}\to Yz \qquad (7)$$

From the compatibility of the pair (F,Y) we get

$\lim_{m\to\infty} K(FYa_{2m}, YFa_{2m}, kt)=0$, $\lim_{m\to\infty} L(FYa_{2m}, YFa_m, kt)=1$ or $\lim_{m\to\infty} FYa_{2m}=\lim_{m\to\infty} YFa_{2m}$

Using (7) this gives that $Fz=Yz$.

Since $F(W)\subset Z(W)$ there exists $v\in W$ such that $Fz=Zu$.

we consider

 ISSN: 0012-0413

$$K(Fz,z,kt)=\lim_{n\to\infty}K(Fz,Ga_{2m+1},kt)\geq\lim_{m\to\infty}\left\{\alpha\frac{K(Za_{2m+1},Ga_{2m+1},t)[1+K(Yz,Fz,t)]}{[1+K(Yz,Za_{2m+1},t)]}+\beta\,K(Yz,Za_{2m+1},t)\right\}$$

$$L(Fz,z,kt)=\lim_{m\to\infty}L(Fz,Ga_{2m+1},kt)\leq\lim_{m\to\infty}\left\{\alpha\frac{L(Za_{2m+1},Ga_{2m+1},t)[1+L(Yz,Fz,t)]}{[1+L(Yz,Za_{2m+1},t)]}+\beta\,L(Yz,Za_{2m+1},t)\right\}$$

this gives $K(Fz,z,kt)\geq\beta K(Fz,z,kt)$, $L(Fz,z,kt)\leq\beta L(Fz,z,kt)$ since $\beta\geq 0,\alpha+\beta<1$
giving that $K(Fz,z,kt)=0, L(Fz,z,kt)=0$. Thus $Fz=z.$
Hence $Fz=Yz=z=Zv.$ This shows that 'z' is a stable fixed point of Y and F.
Now we prove $Zv=Gv.$
Consider

$$K(z,Gv,kt)=K(Fz,Gv,kt)\geq\left\{\alpha\frac{K(Zv,Gv,t)[1+K(Yz,Fz,t)]}{[1+K(Yz,Zv,t)]}+\beta\,K(Yz,Zu,t)\right\}$$
$$=\alpha\,K(z,Gv,t)$$
$$L(z,Gv,kt)=L(Fz,Gv,kt)\leq\left\{\alpha\frac{L(Zv,Gv,t)[1+L(Yz,Fz,t)]}{[1+K(Yz,Zv,t)]}+\beta\,N(Yz,Zv,t)\right\}$$
$$=\alpha\,L(z,Gv,t)$$

this gives $K(z,Gv,kt)\geq\alpha\,K(z,Gv,kt)$, $L(z,Gv,kt)\leq\alpha\,L(z,Gv,kt)$ since
$\alpha\geq 0,\alpha+\beta<1$ giving that $K(z,Gv,kt)=0, L(z,Gv,kt)=0.$
Thus $Gv=z.$ Hence $Gv=Zv=z.$
Also meanwhile the pair (G,Z) is inadequately corresponding and since $Gv=Zv=z.$ we get
$GZv=ZGv$ or $Gz=Zz.$
Again we consider

$$K(z,Gz,kt)=K(Fz,Gz,kt)\geq\left\{\beta\frac{K(Zz,Gz,t)[1+K(Yz,Fz,t)]}{[1+K(Yz,Zz,t)]}+\beta K(Yz,Zz,t)\right\}$$
$$¿\beta K(z,Gz,t)$$
$$L(zi,Gz,kt)=L(Fz,Gz,kt)\leq\left\{\beta\frac{L(Zz,Gz,t)[1+L(Yz,Fz,t)]}{[1+K(Yz,Zz,t)]}+\beta L(Yz,Zz,t)\right\}$$
$$¿\beta L(z,Gz,t)$$

this gives $K(z,Gz,kt)\geq\beta\,K(z,Gz,kt)$, $L(z,Gz,kt)\leq\beta\,L(z,Gz,kt)$ since
$\beta\geq 0,\alpha+\beta<1$ giving that $K(z,Gz,kt)=0, K(z,Gz,kt)=0.$ Thus $Gz=z.$
Hence $Zz=Gz=z.$ Therefore $Yz=Zz=Fz=Gz=z,$ showing that ' Z' is a ordinary fixed point of Y, Z, F and G. One can easily prove the distinctiveness of the fixed point.

Remark 3.2. Theorem 3.1 extends Theorem 2.5 by introducing weaker conditions, offering a broader framework for fixed point theory in intuitionistic FM spaces. Instead of requiring the continuity of one of the mappings, it incorporates reciprocal continuity and compatibility between the pair of mappings. Additionally, the theorem replaces the stronger condition of compatibility with weak compatibility, allowing for more flexible conditions. It also introduces an associated sequence linked to four self-maps Y, Z, F, and G, thereby generalizing the original result and eliminating the need for a complete metric space. This approach enhances the applicability of the theorem in more diverse settings.